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Computational Method for Minimax Optimization in the Time Domain

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A computational method is proposed in this paper for a minimax problem in the time domain. A minimax problem for a linear system is formulated and analyzed by vector space methods. The uniqueness of the optimal solution is analyzed readily in the present approach, and its geometric interpretation is presented. The further analysis based on duality shows that the infimum or a lower bound of the value of the performance index is calculated through supplementary optimization problems, and the minimax problem is solved as an optimization problem with an inequality constraint. The proposed approach provides a simple computational method for the minimax problem. Two numerical examples demonstrate simple implementation of the proposed method.

Nomenclature

ImΩ = image of a set Ω
SGN = sign function of a vector, SGN(x) = [sgn(x_i)]
T^* = adjoint operator of a linear operator T
t_f = terminal time, $> t_0$
t_0 = initial time
U = space of the control input, L_∞^m
X^* = dual space of a normed vector space X, i.e., the space of all bounded linear functionals on X
$<x, x^*>$ = value of a functional $x^* \in X^*$ at $x \in X$; the norm of x^* is defined by $\|x^*\| = \sup_{\|x\| \le 1} |<x, x^*>|$
Y = space of the output, L_∞^r
Ω = family of admissible controls, $\subset L_\infty^m$
$\|\cdot\|$ = norm of a point in a normed vector space
$(\cdot)_i$ = ith component of a vector or ith row of a matrix
$|\cdot|_p$ = l_p norm of a vector, $|x|_p = \left(\sum_i |x_i|^p\right)^{1/p}$
$|\cdot|_\infty$ = l_∞ norm of a vector, $|x|_\infty = \max_i |x_i|$
$|\cdot|_1$ = matrix norm induced by l_∞ norm, $|A|_1 = \max_{x \ne 0} \left(\frac{|Ax|_\infty}{|x|_\infty}\right) = \max_i \sum_j |a_{ij}|$, $[A = (a_{ij})]$
$\|\cdot\|_1$ = L_1 norm of a vector-valued function, $\|x\|_1 = \int_{t_0}^{t_f} |x(t)|_1 \, \mathrm{d}t$
$\|\cdot\|_2$ = L_2 norm of a vector-valued function, $\|x\|_2 = \left[\int_{t_0}^{t_f} x^T(t)x(t) \, \mathrm{d}t\right]^{1/2}$
$\|\cdot\|_\infty$ = L_∞ norm of a vector-valued function, $\|x\|_\infty = \operatorname{ess\,sup}_{t_0 \le t \le t_f} |x(t)|_\infty$

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I. Introduction

MINIMAX optimization is a direct approach to satisfy prescribed design specifications in control design. The H_∞ control theory[1] is one of the minimax optimization theories and shapes the frequency responses of linear feedback systems. The L_1 control theory[2,3] minimizes the maximum amplitude of the system error when the disturbance to the system is unknown but bounded in amplitude. The set-theoretic control synthesis technique[4] maximizes the amplitude of the disturbance that the system can tolerate without violating certain predetermined constraints and can be formulated as an application of the L_1 optimal control theory.[5] Those theories treat linear feedback systems and are mainly intended to reject disturbances.

In contrast to the aforementioned theories for linear feedback systems, the minimax problem to be studied in this paper treats the terminal control, and the objective of the optimization is rather to shape transient responses of a system than to reject disturbances. The minimax problem for terminal control is often called the Chebyshev minimax optimal control problem and has been studied during past years. Johnson[6] discussed geometric properties of the minimax solution in the state space. Barry[7] gave a Mayer-type formulation of the problem and proposed an approximation method to yield a suboptimal minimax control that is arbitrarily close and in many cases identical to the optimal control. Michael[8] demonstrated a general method for the efficient computation of the minimax optimal control for nonlinear systems. In Refs. 9–13, the minimax problem is formulated as a state variable inequality constraint with a parameter to be optimized, and necessary conditions of the optimal control are derived. Although formulation and solution of the minimax problem have been intensively studied in previous work, its application is restricted mainly to trajectory optimization[8,11,12,14] in aerospace engineering.

The authors applied the minimax problem to the shaping of time responses of a generic dynamical system.[15,16] The maximum magnitude of the weighted output is minimized by the control input of a prescribed class. The weighting matrix of the output vector is dependent on time to prescribe the decay rate of the output. The control performance is expressed explicitly by the design parameters and the value of the performance index. This type of minimax problem is applied in Ref. 16 to the slew maneuver of a flexible beam, which has application to the attitude control of a spacecraft with a flexible appendage and has not been treated in previous work on minimax problems. As the result of a numerical simulation and a hardware experiment, the minimax problem is shown to be suitable to make a reasonable tradeoff between the settling time of the attitude angle and the maximum magnitude of the bending moment at the root of the flexible beam.

In this paper, a minimax problem is considered in a rather restricted situation, i.e., the shaping of time responses for a linear system. The linearity assumption facilitates the analysis of the problem.[17] Analysis in Ref. 17 corresponds to a case with a scalar output, whereas the present analysis treats a system with multiple outputs. First, it is shown that time responses of a linear system can be shaped by a minimax optimal control problem. Then the minimax problem is analyzed in terms of normed vector spaces. In comparison with most analysis in the conventional work that is based on the calculus of variation or the minimum principle, the present analysis in normed vector spaces shows another aspect of the problem and provides intuitive geometric interpretations of the problem. It is shown readily that the optimal solution of the minimax problem may not be unique. Therefore the minimax problem is reformulated as minimization of a secondary performance index subject to an inequality constraint so that the uniqueness of the solution is guaranteed. When the minimax problem is solved as an optimization problem with inequality constraint, the minimum of the performance index has to be known previously or searched through computation. This paper shows that the infimum or a lower bound of the performance index is calculated by solving supplementary optimization problems. If the infimum or a lower bound of the performance index is obtained, they can be utilized in solving the minimax problem numerically. This paper proposes a computational method that employs the penalty method to deal with the inequality constraint whose bound is given by the known infimum or lower bound of the performance index. The proposed method yields a near-optimal solution even if the lower bound is so optimistic that it cannot be achieved by any admissible control. Two numerical examples demonstrate that the proposed approach is practically useful, making a reasonable tradeoff between computational efforts and the accuracy of the solution.

II. Analysis of a Minimax Problem in Vector Spaces

A. Problem Formulation

We analyze the shaping of time responses for a linear system, which is an application of the minimax optimal control problem. The system is described by a state equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t); \qquad x(t_0) = x_0 \tag{1}$$

$$y(t) = C(t)x(t) \tag{2}$$

where $x(t) \in \boldsymbol{R}^n$ denotes the state, $u(t) \in \boldsymbol{R}^m$ the control input, and $y(t) \in \boldsymbol{R}^r$ the output to be controlled. The state at the given initial time t_0 is denoted by x_0. The control input is assumed to belong to a family of admissible controls denoted by Ω. Given a reference output $y_{\mathrm{ref}}(t)$, we consider a minimax problem that minimizes the following objective function to shape the time responses of the previous system:

$$J = \operatorname*{ess\,sup}_{t_0 \le t \le t_f} |W(t)[y_{\mathrm{ref}}(t) - y(t)]|_\infty \tag{3}$$

where $W(t)$ is a time-dependent weighting matrix to specify the settling property of the output, and t_f is the given terminal time. The reference $y_{\mathrm{ref}}(t)$ may be omitted if the output is controlled to converge zero from a nonzero initial state. The magnitude of the elements of the output vector are bounded as

$$|y_{\mathrm{ref}\,i}(t) - y_i(t)| \le J\,|W^{-1}(t)|_1 \qquad \text{a.e.} \tag{4}$$

where the weighting matrix is assumed to be nonsingular. Note that the time response of an output is specified by the performance index and the weighting matrix explicitly. The value of $|W^{-1}(t)|_1$ has to decrease as the time increases so that the error $y_{\mathrm{ref}\,i}(t) - y_i(t)$ converges to zero. Especially, if the weighting matrix is given as a diagonal matrix,

$$W(t) = \mathrm{diag}\,(w_1 e^{\beta_1 t}, \ldots, w_r e^{\beta_r t}), \qquad (w_i > 0;\ i = 1, 2, \ldots, r) \tag{5}$$

then the output is bounded by a more strict bound than Eq. (4) as

$$|y_{\mathrm{ref}\,i}(t) - y_i(t)| \le J\,e^{-\beta_i t}/w_i \qquad \text{a.e.} \tag{6}$$

The parameters w_i are scaling factors to evaluate all outputs in the same unit, and the parameters β_i specify the convergent property of the output. Those design parameters are chosen taking the settling time and the acceptable maximum value of the output into account.[16]

The present minimax problem can also be formulated as a minimum norm problem in a vector space as follows:

$$\inf_{u \in \Omega} \|y_0 - Tu\|_\infty \tag{7}$$

where

$$y_0(t) = W(t)[y_{\mathrm{ref}}(t) - C(t)\Phi(t, t_0)x_0] \tag{8}$$

$$(Tu)(t) = \int_{t_0}^{t} W(t)C(t)\Phi(t, \tau)B(\tau)u(\tau)\,\mathrm{d}\tau \tag{9}$$

and $\Phi(t, \tau)$ denotes the transition matrix of the linear system. A control input u is mapped by a bounded linear operator T from the space of the control input $U = L_\infty^m$ to the space of the output $Y = L_\infty^r$. The minimum norm problem is a problem of approximating $y_0 \in Y$ by Tu with $u \in \Omega \subset U$ that minimizes the error in the sense of the L_∞ norm. We restrict the family of admissible controls Ω to the form

$$\Omega = \{u : \|u\|_\infty \le \gamma\}, \qquad (\gamma > 0) \tag{10}$$

which is a convex subset in U. Note that the image of Ω by T, $\mathrm{Im}\Omega$, is also convex, since Ω is convex and T is a bounded linear operator.

B. Uniqueness of Optimal Solution

Other classes of optimal control problems can also be formulated by employing other norms in Eq. (7), which results in other criteria. The following definition and theorem distinguish the minimax criterion from other criteria.

Definition: A norm on a normed vector space X is called strictly convex if the following holds

$$x, y \in X, \qquad x \ne y, \qquad \|x\| = \|y\| = 1 \Rightarrow \|x + y\| < 2 \tag{11}$$

Theorem 1: Let X and W be normed vector spaces, and let the norm on X be strictly convex. Let a bounded linear operator $A{:}W \to X$ be one-to-one, and let K be such a convex set that $K \subset W$. For an $x \in X$, if there exists some $w_0 \in K$ such that

$$\|x - Aw_0\| = \inf_{w \in K} \|x - Aw\| \tag{12}$$

then w_0 is unique.

Proof: If there are $w_1, w_2 \in K$, such that $w_1 \ne w_2$ and

$$\|x - Aw_1\| = \|x - Aw_2\| = \inf_{w \in K} \|x - Aw\| \tag{13}$$

then $(w_1 + w_2)/2 \in K$ because K is a convex set, and $x - Aw_1 \ne x - Aw_2$ because A is one-to-one. Therefore we obtain the following inequality:

$$\inf_{w \in K} \|x - Aw\| \leq \|x - A[(w_1 + w_2)/2]\|$$
$$= \|(x - Aw_1)/2 + (x - Aw_2)/2\| \quad (14)$$
$$< \inf_{w \in K} \|x - Aw\|$$

which is contradictory. □

A geometric interpretation of the previous theorem is illustrated in Fig. 1. If the norm is the l_2 norm, w_0 is the unique point of contract between the circle with its center at x and the line that represents ImK. In the l_∞ norm case, w_0 is any point in the segment where the square contacts the line. It is evident that the l_2 norm is strictly convex and the l_∞ norm is not strictly convex. Theorem 1 implies that the optimal solution of the minimax problem is not guaranteed to be unique, because the L_∞ norm is not strictly convex, in contrast to cases of quadratic criteria with the L_2 norm that is strictly convex.

Although the optimal solution may not be unique in the minimax problem, if one obtains the infimum of the performance index of the minimax problem,

$$d = \inf_{u \in \Omega} \|y_0 - Tu\|_\infty \quad (15)$$

then a solution of the problem is obtained by solving an optimal control problem to minimize a certain performance index

$$J_2 = \int_{t_0}^{t_f} L[x(t), u(t), t]\, \mathrm{d}t \quad (16)$$

subject to an inequality constraint

$$|(y_0 - Tu)(t)|_\infty \leq d \quad (17)$$

The inequality constraint Eq. (17) requires the solution to be an optimal solution for the minimax problem, and the optimality condition for the performance index Eq. (16) is introduced to determine the optimal solution uniquely.

To analyze the previous idea in vector spaces, we define a set $M_\gamma \subset U$ as follows:

$$M_\gamma = \{u : \|y_0 - Tu\|_\infty \leq \gamma\} \quad (18)$$

The set M_γ is convex since

$$u_1, u_2 \in M_\gamma, 0 \leq \alpha \leq 1 \Rightarrow \|y_0 - T[\alpha u_1 + (1 - \alpha)u_2]\|_\infty$$
$$= \|\alpha(y_0 - Tu_1) + (1 - \alpha)(y_0 - Tu_2)\|_\infty$$
$$\leq \alpha\|y_0 - Tu_1\|_\infty + (1 - \alpha)\|y_0 - Tu_2\|_\infty \leq \gamma \quad (19)$$

Therefore $\Omega \cap M_\gamma$ is also convex. Next, we define a linear operator $S : U \to Y \times U$ as

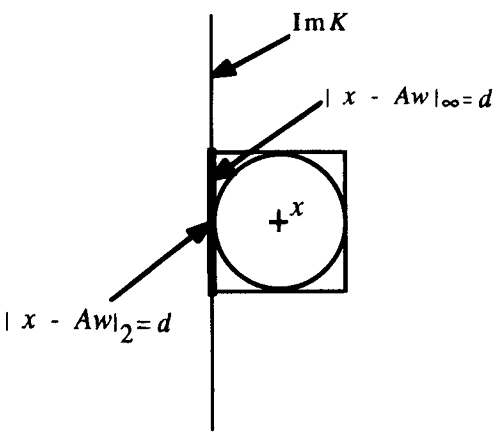

Fig. 1 Geometric interpretation of Theorem 1.

$$S = \begin{bmatrix} T \\ 1_U \end{bmatrix} \quad (20)$$

where 1_U denotes the identity mapping in the space of the control input U. The operator S is one-to-one even if T is not one-to-one, because $Su = 0$ always implies $u = 0$. The identity mapping 1_U does not have to be introduced when the operator T is one-to-one by itself. The operator T is one-to-one if and only if it is left invertible, i.e., T^* is right invertible. The necessary and sufficient condition for a linear system to be right invertible is summarized in Ref. 18 for time-invariant systems, which is not the main topic of this paper and is not further mentioned here. If the norm $\|\cdot\|$ on $Y \times U$ is strictly convex, Theorem 1 implies that if there exists some $u_0 \in \Omega \cap M_\gamma$ such that

$$\left\| \begin{bmatrix} y_0 \\ 0 \end{bmatrix} - Su_0 \right\| = \inf_{u \in \Omega \cap M_\gamma} \left\| \begin{bmatrix} y_0 \\ 0 \end{bmatrix} - Su \right\| \quad (21)$$

then u_0 is unique. If γ is set to be d, $\Omega \cap M_\gamma$ is the set of the minimax optimal control inputs and u_0 is a minimax optimal control input that minimizes the secondary performance index of Eq. (21). The introduction of a secondary performance index is also discussed by Johnson.[6] A geometric interpretation of the present formulation is given in Fig. 1 again. The set $\Omega \cap M_\gamma$ is the inverse image of the segment where the square contacts the line. Minimization of the l_2 norm over $\Omega \cap M_\gamma$ has a unique solution, i.e., the inverse image of the point of contact between the circle and the line.

C. Computational Method Using Infimum or Lower Bound of Performance Index

When the minimax problem is solved as an optimization problem with an inequality constraint, the minimum of the performance index has to be known previously or searched through computation. It is desirable to obtain the minimum or its estimates before computation to reduce the computational efforts. Some fundamental results[19] in vector space methods are available to obtain the infimum of the performance index in the minimax problem.

Definition: Let X be a normed vector space, and let X^* be the dual space of X. A vector $x^* \in X^*$ is said to be aligned with a vector $x \in X$ if $\langle x, x^* \rangle = \|x^*\|\,\|x\|$.

Definition: Let K be a convex set in a real normed vector space X. The functional

$$h(x^*) = \sup_{x \in K} \langle x, x^* \rangle \quad (22)$$

defined on X^* is called the support functional of K.

Theorem 2: Let x_0 be a point in a real normed vector space X, and let $d > 0$ denote its distance from the convex set K having support functional h; then

$$d = \inf_{x \in K} \|x_0 - x\| = \max_{\|x^*\| \leq 1} [\langle x_0, x^* \rangle - h(x^*)] \quad (23)$$

where the maximum on the right is achieved by some $x_1^* \in X^*$. If the infimum on the left is achieved by some $x_1 \in X$, then x_1^* is aligned with $x_0 - x_1$.

Proof: Since the proof is rather lengthy and needs further preparations, it is omitted here. See Theorem 1 of Sec. 5.13 in Ref. 19. □

Geometric interpretations of the previous definitions and theorem are found in Ref. 19. We apply Theorem 2 to the minimum norm problem for the minimax problem with $X = Y$ and $K = \mathrm{Im}\Omega$. The assumptions in the theorem are satisfied by the present setup of the minimax problem. The support functional $h(y^*)$ is derived at first. We have

$$h(y^*) = \sup_{y \in \mathrm{Im}\Omega} <y, y^*> = \sup_{u \in \Omega} <Tu, y^*>$$
$$= \sup_{u \in \Omega} <u, T^*y^*> = \gamma \|T^*y^*\|_1 \tag{24}$$

where $T^* : Y^* \to U^*$ denotes the adjoint operator of T, i.e.,

$$(T^*y^*)(t) = \int_t^{t_f} B^T(t)\, \Phi^T(\tau, t) C^T(\tau) W^T(\tau) y^*(\tau)\, d\tau \tag{25}$$

The dual spaces Y^* and U^* are direct products of L_∞^*. The dual space L_∞^* consists of real-valued bounded additive set functions that vanish on sets of measure zero, and the norm of an element in L_∞^* is its total variation.[20] We regard L_1 as L_∞^* for simplicity without affecting the result of the present analysis, although L_1 is identified with a subspace of L_∞^*, not the whole L_∞^* in the strict sense. The last equality in Eq. (24) is achieved by $u_1 \in \Omega$ given as

$$u_1(t) = \gamma \mathrm{SGN}[(T^*y^*)(t)] \tag{26}$$

Now the infimum of the performance index of the minimax problem is expressed as follows:

$$d = \max_{\|y^*\|_1 \le 1} [<y_0, y^*> - \gamma \|T^*y^*\|_1] \tag{27}$$

Although the right-hand side of Eq. (27) cannot be evaluated in a straightforward manner, further results are obtained by analyzing a more restricted case. Suppose, for the optimal control input u_1, the maximum error of the output occurs at only $t = t_1 \in [t_0, t_f]$ and in only the ith component, i.e.,

$$\|y_0 - Tu_1\|_\infty = |(y_0 - Tu_1)_i(t_1)| \tag{28}$$

holds for only one combination of t_1 and i. Then the alignment condition in Theorem 2 implies that $y^* \in Y^*$, which achieves the maximum in Eq. (27), must be in the form

$$y^*(t) = \begin{cases} 0, & t < t_1 \\ \mathrm{sgn}[(y_0 - Tu_1)_i(t_1)]e_i, & t \ge t_1 \end{cases} \tag{29}$$

where e_i denotes a vector with only the ith component as 1 and other components as zero. The optimal control input $u_1(t)$ is determined from Eq. (26) and is expressed as

$$u_1(t) = -\gamma \mathrm{SGN}[B^T(t)\lambda(t)] \tag{30}$$

where $\lambda(t)$ is governed by the following differential equation in $[t_0, t_1]$,

$$\dot{\lambda}(t) = -A^T(t)\lambda(t)$$
$$\mathrm{gl}(t_1) = -\mathrm{sgn}[(y_0 - Tu_1)_i(t_1)][C^T(t_1)W^T(t_1)]_i \tag{31}$$

and is a null vector in $[t_1, t_f]$. Since y^* is fixed to the form in Eq. (29) under the assumption, and only t_1 and i are unknown, the infimum of the performance index is expressed as

$$d = \operatorname*{ess\,sup}_{t_0 \le t_1 \le t_f, i} |(y_0 - Tu_1)_i(t_1)| \tag{32}$$

As seen from Eqs. (30) and (31), $\lambda(t)$ corresponds to a costate in the minimum principle, and $u_1(t)$ and $\lambda(t)$ satisfy the necessary condition to minimize a performance index

$$J = |(y_0 - Tu_1)_i(t_1)| \tag{33}$$

Minimization of the previous performance index is equivalent to minimization of the following performance index:

$$J' = \{W(t_1)[y_{\mathrm{ref}}(t_1) - y(t_1)]\}_i^2 \tag{34}$$

The infimum of the performance index is evaluated by solving the optimal control problem minimizing the performance indices Eqs. (33) or (34), varying t_1 and i in sequence, and by searching the maximum of the optimal performance index with respect to t_1 and i, i.e.,

$$d = \inf_{u \in \Omega} \|y_0 - Tu\|_\infty = \operatorname*{ess\,sup}_{t_0 \le t_1 \le t_f, i} [\min_{u \in \Omega} |(y_0 - Tu)_i(t)|] \tag{35}$$

The optimal control problems with the performance indices Eqs. (33) or (34) are simple optimal control problems with terminal penalties and can be solved more readily by numerical methods than the original minimax problem. In practice, the time t_1 on the right-hand side of Eq. (35) needs to be searched over a smaller interval than the whole interval, because it is often the case that the maximum magnitude of the output occurs soon after the initial time.

The assumptions made are so restricting that the previous results do not apply to the broad class of minimax optimization problems. Next we discuss a relationship that is similar to Eq. (35) for the general minimax problem.

Theorem 3[21]: Let Ω and I be nonempty sets. For a function $f: I \times \Omega \to [-\infty, \infty]$, the following inequality holds:

$$\sup_{t \in I} \left[\inf_{u \in \Omega} f(t, u) \right] \le \inf_{u \in \Omega} \left[\sup_{t \in I} f(t, u) \right] \tag{36}$$

Proof: For any combination of $u_1 \in \Omega$ and $t_1 \in I$, the following holds:

$$\inf_{u \in \Omega} f(t_1, u) \le f(t_1, u_1) \le \sup_{t \in I} f(t, u_1) \tag{37}$$

Therefore

$$\sup_{t \in I} \left[\inf_{u \in \Omega} f(t, u) \right] \le \sup_{t \in I} f(t, u_1) \tag{38}$$

holds for any $u_1 \in \Omega$, and then inequality (36) holds. □

The right-hand side of the inequality (36) is identical with the infimum of the performance index in a minimax problem, when Ω is a family of admissible control, I is the time interval of control, and f is a functional the supremum of which is to be minimized. Theorem 3 is simpler and can deal with more general cases than the results for linear systems. However, it gives a lower bound, not the infimum, of the performance index. If there is a control input that achieves the lower bound, the control input is an optimal solution. If not, one has to invoke other approaches to calculate the optimal solution. One can employ a penalty method to obtain a near-optimal solution even if the lower bound cannot be achieved. Denoting the lower bound of the inequality (36) by d_0, a penalty method for the inequality constraint

$$f(t, u) \le d_0 \tag{39}$$

yields an optimal solution of enough degree of accuracy if the lower bound can be achieved and minimizes the violation of the constraint in a certain sense even if the lower bound cannot be achieved, resulting in a near-optimal solution. Therefore the penalty method is a reasonable technique to solve the minimax problem utilizing the infimum or a lower bound of the performance index.

The proposed computational method is summarized as follows:

1) Solve $\min_{u \in \Omega} |(y_0 - Tu)_i(t_1)|$ and search its maximum d_0 with respect to t_1 and i. The maximum gives the infimum or a lower bound of the performance index of the original minimax problem. Note that t_1 may not have to be searched over the

whole interval, since the maximum occurs near the initial time in many cases.

2) Solve an optimal control problem that minimizes a given secondary performance index subject to an inequality constraint, $\|(y_0 - Tu)(t)\|_\infty \leq d_0$. The penalty method yields a near-optimal solution even if the lower bound is too small so that any control input cannot satisfy the constraint.

III. Numerical Examples

A. Second-Order System

The first example is the shaping of the output response of a second-order system that consists of a mass and a spring (Fig. 2). The spring has unit deflection and zero velocity at the initial time. The output $y(t)$ to be controlled is the deflection $v(t)$ of the spring. The performance index is given by

$$J = \max_{0 \leq t \leq 10} |W(t)y(t)|, \qquad W(t) = e^{0.5t} \tag{40}$$

where the output is weighted by an exponential function so as to converge to zero. The control input is a force applied to the mass, and the family of admissible controls Ω in this case is

$$\Omega = \{u(t) : |u(t)| \leq 1\} \tag{41}$$

The infimum of the performance index is obtained as $d = 1.06$ through use of Eq. (35). The maximum magnitude of the output is also estimated to occur at $t_1 = 0.24$. A penalty function of the form

$$J_P = \int_0^{10} P[\sigma(t)]\,\mathrm{d}t \tag{42}$$

is employed with

$$P[\sigma(t)] = \begin{cases} 0, & \sigma(t) < 0 \\ \cosh[c\sigma(t)] - 1 \ (c > 0), & \sigma(t) \geq 0 \end{cases} \tag{43}$$

$$\sigma(t) = [W(t)y(t)]^2/2 - d^2/2 \tag{44}$$

The penalty function is chosen so that it has the continuous first-order derivative and it increases exponentially when the constraint is violated. The exponential penalty function is expected to result in a more accurate solution than a simple quadratic penalty function. The secondary performance index is introduced to guarantee the uniqueness of the optimal solution and is selected as

$$J_2 = \left\| \begin{bmatrix} W(t)y(t) \\ u(t) \end{bmatrix} \right\|_2 \tag{45}$$

which is the L_2 norm on $Y \times U$ and is strictly convex.

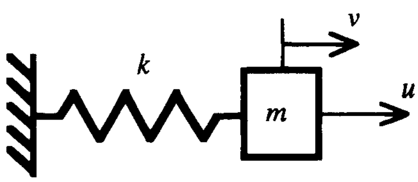

Fig. 2 Second-order system.

Figure 3 shows the resultant responses of the output $y(t)$, the control input $u(t)$, the weighted norm $|e^{0.5t}y(t)|$ (broken line), and $z(t)$ (solid line) defined as

$$z(t) = \max_{0 \leq \tau \leq t} |W(\tau)y(\tau)| \tag{46}$$

The clipping-off conjugate gradient algorithm[22] is employed as the numerical method to solve the present optimization problem. The expected value of the performance index is achieved, and the output response is successfully shaped by the minimax problem. The maximum magnitude of the weighted output occurs at $t = 0.24$ as expected, and the secondary performance index is minimized during the interval [0.24, 10]. The control input saturates in [0, 0.24] according to Eq. (26). Minimization of a proper secondary performance index generates a smooth time response after the maximum magnitude occurs.

B. Slew Maneuver

The second example employs the slew maneuver of a flexible beam attached to a rigid hub (Fig. 4), which has application to

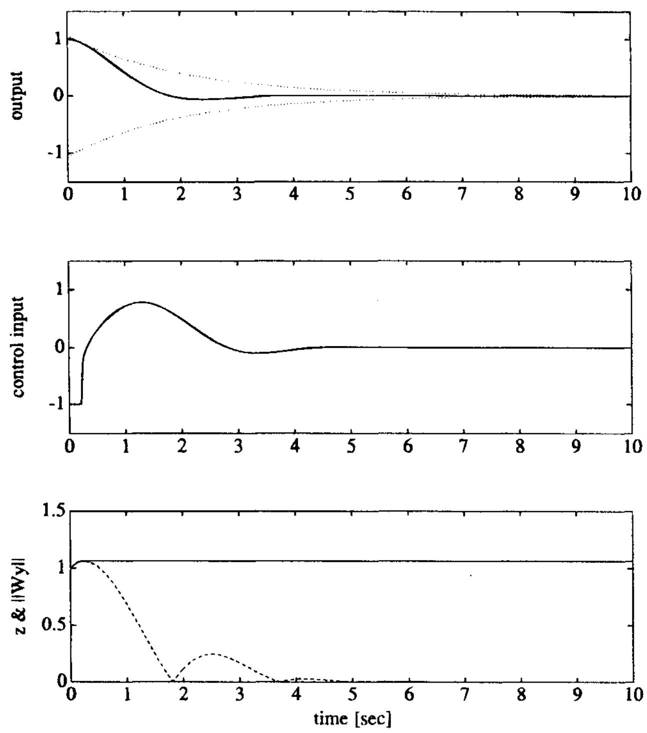

Fig. 3 Numerical solution of the minimax problem for the second-order system.

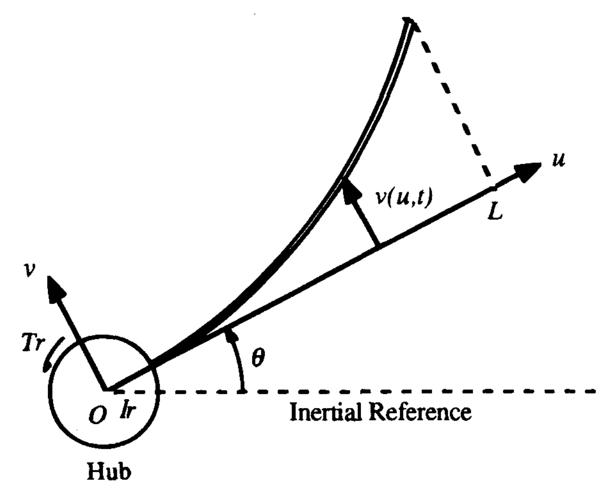

Fig. 4 Slew maneuver model.

attitude control of a spacecraft with a flexible appendage. The model is also treated numerically and experimentally in Ref. 16. The control objective is to change the attitude angle of the rigid hub with the minimal vibration excitation of the flexible beam. The output variables to be controlled are the hub angle $\theta(t)$ and the bending moment at the root of the beam $M_0(t)$, i.e.,

$$y(t) = [\theta(t) \quad M_0(t)]^T \tag{47}$$

The hub angle is equal to -1.05 rad (-60 deg), and the flexible beam is at the equilibrium position at the initial time. The performance index for this case is given as follows:

$$J = \max_{0 \le t \le 10} |W(t)y(t)|_\infty \tag{48}$$

where $W(t)$ is a weighting matrix defined as

$$W(t) = e^{0.5t}\begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix} \tag{49}$$

The control input is the torque $T_r(t)$ in the rigid hub, and the family of admissible controls is

$$\Omega = \{T_r(t) : |T_r(t)| \le 0.05\} \tag{50}$$

Because of the continuity condition required by the computational method, the l_∞ norm is approximated by the l_{80} norm in the numerical computation. The error of this approximation is less than 1% for two-dimensional vectors, since $|x|_\infty \le |x|_{80} \le 2^{1/80}|x|_\infty$ ($2^{1/80} < 1.01$) holds. The lower bound of the performance index is obtained as $d_0 = 1.15$. A penalty function and a secondary performance index are introduced in a similar form to Eqs. (42–45). Figure 5 presents a solution calculated by a conventional approximate method,[8,16] and Fig. 6 presents a result obtained by the proposed method. In the conventional approximate method, the control input minimizing the performance index

$$J'_p = \left\{\int_0^{10} |W(t)y(t)|_{80}^p \, dt\right\}^{1/p}, \qquad (p > 0) \tag{51}$$

is given as the optimal control input. The performance index Eq. (48) is well approximated by Eq. (51) with a large enough p, since the following holds:

$$\max_{0 \le t \le 10} |W(t)y(t)|_{80} = \lim_{p \to \infty} J'_p \tag{52}$$

Note that the l_∞ norm is approximated by the l_{80} norm again because of the continuity condition required by the computational method. The parameter p is set at 8 in the computation on the basis of the numeric limits in digital computers. The resultant value of the performance index, Eq. (48), is equal to 1.33 in the conventional method and is equal to 1.16 in the proposed method. The proposed approach yields a satisfactory solution with the simple penalty method, though the resultant value of the performance index is slightly larger than the estimated lower bound. It may be concluded that the proposed method yields a more precise solution than the conventional method even if the lower bound d_0 is not achieved by an admissible control.

IV. Summary

A minimax optimal control problem is formulated and analyzed in normed vector spaces. The problem has applications in shaping of time responses of linear systems. The uniqueness of the optimal solution is discussed, and a computational method for the minimax problem is proposed. The proposed method evaluates the infimum or a lower bound of the performance index and formulates the minimax problem as an optimal control problem with an inequality constraint. In restricted cases, the exact value of the minimum of the performance index is evaluated through solving a sequence of supplementary optimiza-

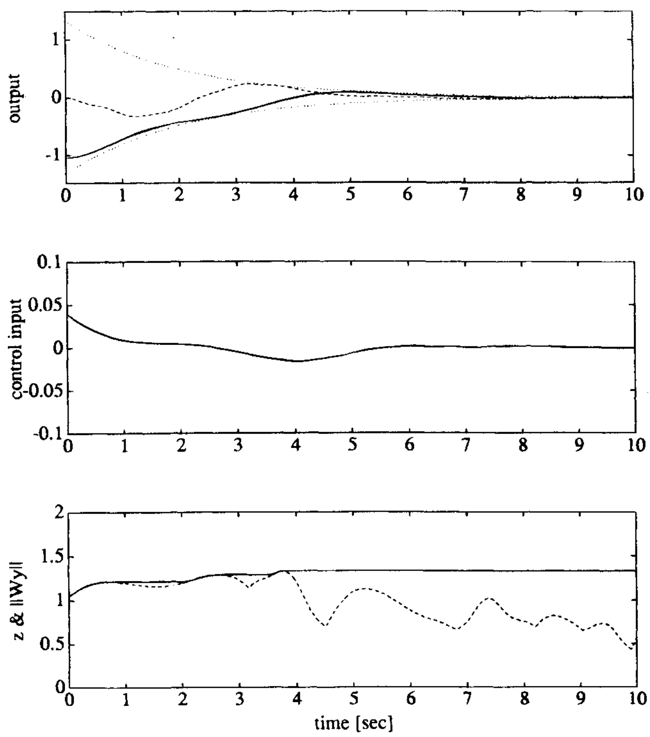

Fig. 5 Numerical solution of the minimax problem for slew maneuver: conventional method.

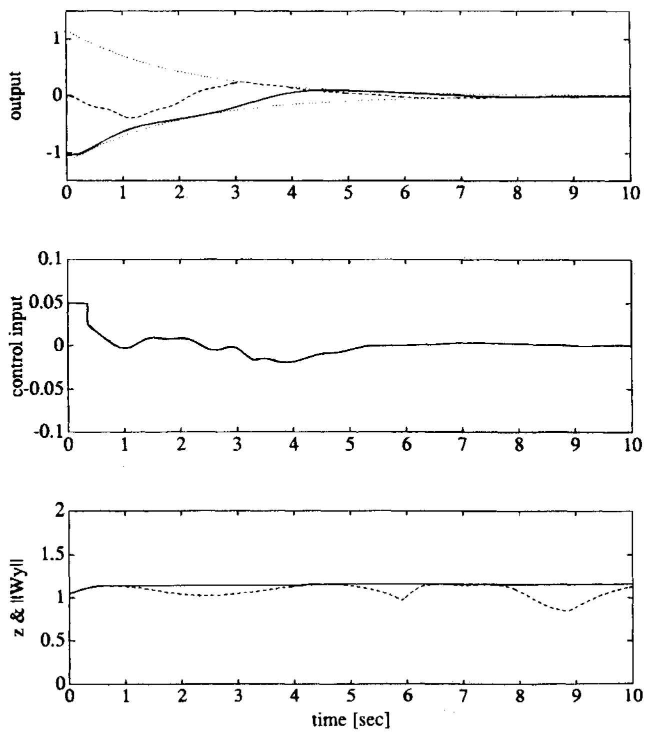

Fig. 6 Numerical solution of the minimax problem for slew maneuver: proposed method.

tion problems. For general cases, only a lower bound of the performance index can be obtained. The penalty method is suitable to deal with the inequality constraint, because it yields a near-optimal solution even if the lower bound is so optimistic that it cannot be achieved by any admissible control. Two numerical examples show that the proposed approach yields solutions of sufficient accuracy with a simple computational method and validates the proposed approach.

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